Lösungen Übungsb 4 6

zu 49) es soll bewiesen werden

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

$$\int (\tan^n x + \tan^{n-2} x)\,dx = \frac{1}{n-1}\tan^{n-1}$$

es wird nun die linke Seite umgeformt:

$$\int (\tan^n x + \tan^{n-2} x)\,dx$$

$$= \int \tan^{n-2} x\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)dx$$

$$= \int \tan^{n-2} x \cdot \frac{1}{\cos^2 x}\,dx \qquad u = \tan x \qquad \frac{du}{dx} = \frac{1}{\cos^2 x}$$

$$= \int u^{n-2} \cdot \frac{1}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}\,du$$

$$= \int u^{n-2}\,du = \frac{u^{n-1}}{n-1} = \frac{1}{n-1}\tan^{n-1} x$$

q.e.d.

$$\int \cos^4 x\,dx = \frac{1}{4}\sin x \cos^3 x\,dx + \frac{3}{4}\int \cos^2 x\,dx$$

$$= \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\left[\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1\,dx\right]$$

$$= \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x \Big|_0^{\pi/4}$$

$$\int_0^{\pi/4} \cos^4 x\,dx = \frac{1}{4}\cdot\left(\frac{1}{2}\sqrt{2}\right)^4 + \frac{3}{8}\left(\frac{1}{2}\sqrt{2}\right)^2 + \frac{3\pi}{32}$$

$$\int_0^{\pi/4} \cos^4 x\,dx = \frac{1}{4} + \frac{3\pi}{32}$$

$$\int_0^{\pi/2} \cos^4 x\,dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x \Big|_0^{\pi/2}$$

$$= \frac{3}{16}\pi$$

$$\int_0^{\pi/2} \sin^8 x\,dx = -\frac{1}{8}\sin^7 x \cos x \Big|_0^{\pi/2} + \frac{7}{8}\int_0^{\pi/2} \sin^6 x\,dx$$

$$= +\frac{7}{8}\int_0^{\pi/2} \sin^6 x\,dx \quad =$$

$$\frac{7}{8}\left[-\frac{1}{6}\sin^5 x \cos x \Big|_0^{\pi/2} + \frac{5}{6}\int_0^{\pi/2} \sin^4 x\,dx\right]$$

$$= \frac{7\cdot 5}{8\cdot 6}\int_0^{\pi/2} \sin^4 x\,dx$$

$$= \frac{7\cdot 5}{8\cdot 6}\left[-\frac{1}{4}\sin^3 x \cos x \Big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \sin^2 x\,dx\right]$$

$$= \frac{7\cdot 5\cdot 3}{8\cdot 6\cdot 4}\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{7\cdot 5\cdot 3}{8\cdot 6\cdot 4}\left[-\frac{1}{2}\sin x \cdot \cos x \Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} 1\,dx\right]$$

$$= \frac{7\cdot 5\cdot 3}{8\cdot 6\cdot 4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{7\cdot 5\cdot 3}{8\cdot 6\cdot 4}\cdot\frac{\pi}{4}$$

$$= \frac{35\pi}{256}$$

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$$

$$= \frac{1}{4}\tan^4 x - \left[\frac{1}{2}\tan^2 x - \int \tan x\,dx\right]$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \int \tan x\,dx$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x|$$

$$\int_0^{\pi/4} \tan^5 x\,dx = \frac{1}{4} - \frac{1}{2} - \ln\frac{1}{2}\sqrt{2}$$

$$= -\frac{1}{4} + \frac{1}{2}\ln 2$$

50) a) $\displaystyle\int \frac{x}{\sqrt{(x^2+25)^3}}\,dx \qquad u = x^2+25 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$

$$= \int \frac{\cancel{x}}{u^{3/2}}\cdot\frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int u^{-3/2}\,du = \frac{1}{2}\,\frac{u^{-1/2}}{-\frac{1}{2}}$$

$$= -\frac{1}{\sqrt{x^2+25}}$$

b) $\displaystyle\int \frac{x^3}{3x+5}\,dx$

$$\frac{x^3}{3x+5} = \frac{1}{3}x^2 - \frac{5}{9}x + \frac{25}{27} - \frac{125}{27}\,\frac{1}{3x+5}$$

mit Polynomdivision

$$\int \frac{x^3}{3x+5}\,dx = \frac{1}{9}x^3 - \frac{5}{18}x^2 + \frac{25}{27}x - \frac{125}{81}\ln(3x+5)$$

c) $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

z.B. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin x(1-\cos^2 x)}{\cos x}\,dx$

$$= \int \tan x\,dx - \int \sin x \cos x\,dx$$

$$= -\ln|\cos x| - \frac{1}{2}\sin^2 x$$

oder: $\sin x = u \qquad \frac{du}{dx} = \cos x \qquad dx = \frac{du}{\cos x}$

$$\int \frac{u^3}{\cos x}\cdot\frac{du}{\cos x} \qquad \cos^2 x = 1-\sin^2 x = 1-u^2$$

$$= \int \frac{u^3}{1-u^2}\,du$$

$$= \int\left(-u + \frac{u}{1-u^2}\right)du \qquad \text{Polynomdiv.}$$

$$= -\frac{1}{2}u^2 - \frac{1}{2}\ln|1-u^2|$$

$$= -\frac{1}{2}\sin^2 x - \frac{1}{2}\ln(\cos^2 x)$$

$$= -\frac{1}{2}\sin^2 x - \ln|\cos x|$$

d) $\displaystyle\int \frac{\sin x\cdot\cos x}{\sin^2 x - 3\cos^2 x}\,dx$

Abl. des Nenners: $2\sin x\cdot\cos x + 6\sin x\cos x = 8\sin x\cdot\cos x$

$$\frac{1}{8}\int \frac{8\sin x\cdot\cos x}{\sin^2 x - 3\cos^2 x}\,dx = \frac{1}{8}\int \frac{f'}{f}\,dx$$

$$= \frac{1}{8}\ln|f(x)| = \frac{1}{8}\ln|\sin^2 x - 3\cos^2 x|$$

51 a) mit Substitution

$$\int \frac{2x+1}{(x+1)^2}\,dx = \int \frac{2u-1}{u^2}\,du$$

$u = x+1$, $dx = du$

$$= \int \frac{2}{u}\,du - \int \frac{1}{u^2}\,du$$

$$= 2\ln|u| + \frac{1}{u}$$

$$= 2\ln|x+1| + \frac{1}{x+1}$$

mit Partialbruchzerlegung

$$\frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$A = 2 \qquad B = -1$

$$\int \text{i. } dx = 2\int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

$$= 2\ln|x+1| + \frac{1}{x+1}$$

b) $\displaystyle\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$A = \lim_{x\to 0}\frac{x^2+1}{(x+1)^2} = 1$$

$$C = \lim_{x\to -1}\frac{x^2+1}{x} = -2$$

B berechnen, indem man z.B. $x = 1$ einsetzt.

$$\frac{2}{1\cdot 4} = \frac{1}{1} + \frac{B}{2} - \frac{2}{4} \rightsquigarrow B = 0$$

$$\int \text{i. } dx = \int \frac{1}{x}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x| + 2\,\frac{1}{x+1}$$